An Improved Secant Type Method for Solving Nonlinear Optimization Problems

Gouri M Krishna[1], Sreedeep C D[2]

Abstract

This project introduces an improved version of the Modern secant- kind method for solving nonlinear optimization problems. Steffensen's method is used to generate the initial approximation, eliminating the need for second derivative calculations and improving computational efficiency. Numerical results obtained from seven test problems demonstrate that the proposed method converges faster with fewer iterations and requires less computational time compared to Newton's method and the Modern secant-kind method.

AMS Subject Classification: 65K05, 65H05

Keywords: Nonlinear optimization, Newton's method, Secant method, Steffensen's method, Modern secant-kind method.

1 Introduction

Optimization plays a fundamental role in many practical disciplines such as engineering design, economics, data science, machine learning, and operational research, where optimal solutions are required to improve performance, reduce cost, and enhance decision-making processes.

Let $g: \mathbb{R} \to \mathbb{R}$ be a nonlinear function. The optimization of g consists of determining the value of x at which $g(x)$ attains an extremum. This task is equivalent to finding the stationary points of the function, which can be obtained by solving the equation

$$g'(x) = 0.$$

where $g'(x)$ denotes the function's first derivative. Consequently, the problem of function optimization can be reduced to solving a nonlinear equation.

[1]Department of Mathematics, Amrita School of Physical Sciences, Amritapuri, India.
[2]Department of Mathematics, Amrita School of Physical Sciences, Amritapuri, India.

Numerous iterative algorithms have been proposed for solving nonlinear optimization problems [2, 6, 7, 8, 12, 14]. Among these, Newton's method [3, 10] remains one of the most widely used techniques and is described by the iteration

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} \tag{1.1}$$

In recent years, several modifications of this classical method have been developed to enhance its convergence properties [1, 5, 13, 15, 16]. Another commonly employed technique for optimization is the Secant method, which is defined by the recurrence relation

$$x_{n+1} = x_n - \frac{g'(x_n)(x_n - x_{n-1})}{g'(x_n) - g'(x_{n-1})}.$$

Various improved versions of the Secant method have likewise been proposed to further increase its efficiency and robustness [9, 11].

In 2025, a new iterative scheme known as the Modern secant-kind method [10] was proposed for solving nonlinear optimization problems involving single- variable functions when an initial approximation is provided. The iterative formula of this method is given by:

$$x_{n+1} = x_n - \frac{g'(x_n)(x_n - x_{n-1})}{\left(\frac{g(x_n) - g(x_{n-1})}{x_{n-1} - x_n}\right) - \frac{1}{2} g'(x_{n-1})} \tag{1.2}$$

Nevertheless, the implementation of this approach requires an additional starting value to initiate the iteration. To obtain this extra initial approximation, Newton's method (1.1) is employed. Since Newton's method relies on higher order derivatives, its use increases the overall computational cost of the procedure.

To address this limitation, we propose an enhanced variant of the Modern secant-kind method [10], referred to as the Steffensen–Secant method. In the process of generating the initial approximation, Steffensen's method [4, 17] is employed. This technique is based on a forward-difference approximation and is defined by:

$$x_n = x_{n-1} - \frac{g'(x_{n-1})^2}{g'\big(x_{n-1} + g'(x_{n-1})\big) - g'(x_{n-1})}. \tag{1.3}$$

The adoption of this strategy removes the need for evaluating second and higher-order derivatives, thereby lowering the overall computational expense. Furthermore, numerical comparisons indicate that the Steffensen–Secant method attains the optimal solution with fewer iterations and reduced execution time than both Newton's method and the Modern secant-kind method.

The remainder of this paper is organized as follows. First, the algorithm of the proposed method is described. Next, numerical experiments are reported, in which the new approach is compared with Newton's method and the Modern secant-kind method [10] using seven benchmark test problems. Finally, the principal results and concluding remarks are presented.

2.Steffensen–Secant Method Algorithm

The Steffensen-Secant algorithm takes as input a function g(x), an initial guess x_0, a tolerance level *tol* and maximum number of iterations *max_iter.* In each iteration, a new value x_1 is computed using Steffensen's method (1.3), which relies on x_0 and the first derivative of the function, $g'(x)$. The algorithm then applies the Modern secant-kind method (1.2), using both x_0 and x_1, to compute a further refined estimate x_2. This process continues until the absolute difference of x_1 and x_2 is less than the specified tolerance, or the maximum number of iterations is reached. Once one of these stopping criteria is met, the latest approximation x_2 is returned as the solution.

Algorithm 1: Steffensen-Secant Algorithm (SS)

Input: Function $g(x)$, initial guess x_0, tolerance tol, maximum iterations max_iter
iter_count ← 0
while iter_count < max_iter **do**
  $〚x〛\_1 \leftarrow x\_0 - 〚(g'(x\_0))〛\hat{}2/(g'(x\_0 + g'(x\_0)) - g'(x\_0))$
  $d \leftarrow (g(x\_1) - g(x\_0))/(x\_0 - x\_1) - 1/2\, g'(x\_1)$
  if $d = 0$ **then**
    return "Division by zero error"
  else
    $(x)_2 \leftarrow x\_1 - (g'(x\_1)(x\_0 - x\_1))/d$
    if $|x\_2 - x\_1| < tol$ **then**
      return x_2, iter_count
    end if
  end if
  $x\_(0) \leftarrow x\_1$
  iter_count ← iter_count + 1
end while
return x_2, iter_count.

3. Numerical Experiments

To compare the performance of Newton's method (NM), the Modern secant-kind method (SK), and the proposed Steffensen–Secant method (SS), a set of benchmark test problems is considered. In the numerical experiments, the iterative procedures are terminated when either the stopping criterion $|x_(n+1) - x_n| < tol$is satisfied, where $tol = 10^{-10}$, or a division-by-zero situation is encountered. The numerical results clearly demonstrate the superiority of the SS method, as it achieves convergence to the optimal solution using fewer iterations and reduced computational time compared to the other two methods.

Here, ***N.I*** denotes the number of iterations, t indicates the computational time in seconds, and x represents the computed optimal value.

Test Problem 1: Initial Approximation $x_0 = 0.25$, Function $g(x) = e^x - 3x^2$.

Method	x	N. I	t
NM	0.2044814493	3	1766298867.0630329
SK	0.2044814493	2	0.007465700000011566
SS	0.2044814493	2	0.005329999999580905

Test Problem 2: Initial Approximation $x_0 = 1$, Function $g(x) = e^{-x} + x^2$.

Method	x	N. I	t
NM	0.3517337112	4	1766298867.0600173
SK	0.3517336983	4	0.009915200000023106
SS	0.3517337109	4	0.003862599999933991

Test Problem 3: Initial Approximation $x_0 = 2$, Function $g(x) = \cos(x) + (x-2)^2$.

Method	x	N. I	t
NM	2.3542427582	4	1766298867.0757813
SK	2.3542427592	4	0.010099000000082015
SS	2.3542427583	3	0.0038974000000280284

Test Problem 4: Initial Approximation $x_0 = -1$, Function $g(x) = e^x + x^2$.

Method	x	N.I	t
NM	-0.3517337112	4	1766298867.0559685
SK	-0.3517336983	4	0.011752900000003315
SS	-0.3517337109	4	0.0069167000000334156

Test Problem 5: Initial Approximation $x_0 = -1$, Function $g(x) = \frac{1}{1+e^{-x}} + x^2$.

Method	x	N.I	t
NM	-0.1245167362	3	1766298867.0795143
SK	-0.1245167362	2	0.016229100000014599
SS	-0.1245167362	2	0.007667099999993509

Test Problem 6: Initial Approximation $x_0 = 1$, Function $g(x) = \tan^{-1}(x) + \sin(x)$.

Method	x	N.I	t
NM	1.8612036163	4	1766298867.0769014
SK	1.8612036163	3	0.011958000000049651
SS	1.8612036163	3	0.010385599999835904

Test Problem 7: Initial Approximation $x_0 = 2$, Function $g(x) = \sin(x) - \frac{x}{2}$.

Method	x	N.I	t
NM	1.0471975511	4	1766298867.0493565
SK	1.0471975511	3	0.006201899999950911
SS	1.0471975511	2	0.005243800000016563

From the above numerical results, it can be observed that for all the test problems considered, the Steffensen–Secant (SS) method consistently requires less computational time than both the Modern secant-kind (SK) and Newton's method (NM). In addition, the number of iterations (N.I) needed by the proposed SS method is either smaller than or equal to that of the SK and NM methods, indicating its superior efficiency and faster convergence behavior.

4. Conclusion

The incorporation of Steffensen's method in the initial approximation stage removes the requirement for evaluating second and higher-order derivatives, which significantly simplifies the computational procedure. By avoiding these derivative calculations, the proposed Steffensen–Secant method reduces the complexity and cost associated with each iteration. Moreover, when compared with Newton's method and the Modern secant-kind method, the Steffensen–Secant scheme demonstrates superior performance by attaining the optimal solution in fewer iterations and with shorter execution time. This improvement leads to a noticeable reduction in the overall computational expense, making the proposed method more efficient and attractive for practical nonlinear optimization problems.

References

[1] Al-Rawi, E. S., Hassan, B. A., and Sulaiman, B. M., A new secant-type method for solving one-variable functions, *Italian Journal of Pure and Applied Mathematics*, 47, 2022.

[2] Argyros, I. K., Computational Theory of Iterative Methods, Elsevier, 2007.

[3] Argyros, I. K., Convergence and Applications of Newton-Type Iterations, Springer, 2008.

[4] Cordero, A., Hueso, J. L., Mart´ınez, E., and Torregrosa, J. R., Steffensen-type methods for solving nonlinear equations, *Journal of Computational and Applied Mathematics*, 236(12), 3058–3064, 2012.

[5] Hassan, B. A., and Al-Rawi, E. S., A modified Newton's method for solving functions of one variable, *Italian Journal of Pure and Applied Mathematics*, 46, 577–582, 2021.

[6] George, S., Sreedeep, C. D., and Argyros, I. K., Secant-type iteration for nonlinear ill-posed equations in Banach spaces, *Journal of Inverse and Ill-posed Problems*, 31(1), 147–157, 2023.

[7] Hassan, M., Ali, M., and Waseem, M., Derivative-free iterative method for solving nonlinear equations with stability analysis, *Southern Journal of Computer Science*, 1(2), 38–63, 2025.

[8] Homeier, H. H., A modified Newton method with cubic convergence: The multivariate case, *Journal of Computational and Applied Mathematics*, 169(1), 161–169, 2004.

[9] Kahya, E., Modified secant-type methods for unconstrained optimization, *Applied Mathematics and Computation*, 181(2), 1349–1356, 2006.

[10] Laylani, Y. A., Abdullah, S. M., and Hassan, B. A., A modern secant-kind

technique with quadratic convergence, *Journal of Interdisciplinary Mathematics*, 28(1), 161–167, 2025.

[11] Salih, Z. F., Hassan, B. A., and Sulaiman, B. M., A new secant-type method with quadratic-order convergence, *European Journal of Pure and Applied Mathematics*, 15(3), 2022.

[12] Soleymani, F., Optimal fourth-order iterative methods free from derivatives, *Miskolc Mathematical Notes*, 12(2), 255–264, 2011.

[13] Sreedeep, C. D., George, S., and Argyros, I. K., Extended Newton-type iteration for nonlinear ill-posed equations in Banach spaces, *Journal of Applied Mathematics and Computing*, 60, 435–453, 2019.

[14] Sreedeep, C. D., Argyros, I. K., Gopika Dinesh, K. C., Shrestha, N., and Argyros, M., Convergence analysis of a power series-based iterative method having seventh-order convergence, *Matematychni Studii*, 64(2), 179–193, 2025.

[15] Sulaiman, B. M., Hassan, B. A., and Sulaiman, R. M., A new secant method for minima in one-variable problems, *Journal of Interdisciplinary Mathematics*, 26(6), 1015–1021, 2023.

[16] Weerakoon, S., and Fernando, T., A variant of Newton's method with accelerated third-order convergence, *Applied Mathematics Letters*, 13(8), 87–93, 2000.

[17] Zafar, F., Cordero, A., Mujtaba, S., and Torregrosa, J. R., A hybrid Steffensen–genetic algorithm for finding multi-roots of nonlinear equations and applications to biomedical engineering, *Algorithms*, 18(9), 582, 2025.